Oscillons and quasi-breathers in D+1 dimensions

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ABSTRACT: We study oscillons in $D+1$ space-time dimensions using a spherically symmetric ansatz. From Gaussian initial conditions, these evolve by emitting radiation, approaching "quasi-breathers", near-periodic solutions to the equations of motion. Using a truncated mode expansion, we numerically determine these quasi-breather solutions in $2 < D < 6$ and the energy dependence on the oscillation frequency. In particular, this energy has a minimum, which in turn depends on the number of spatial dimensions. We study the time evolution and lifetimes of the resulting quasi-breathers, and show how generic oscillons decay into these before disappearing altogether. We comment on the apparent absence of oscillons for $D > 5$ and the possibility of stable solutions for $D \leq 2$.

Contents

1. Introduction

Topological defects such as monopoles, vortex strings and domains walls owe their stability to non-trivial boundary conditions or, equivalently, a conserved (topological) charge [1]. There are other mechanisms which can lead to the formation of lumps in field theory, such as the conservation of a Nöther charge allowing for Q-balls [2] or it may simply be energetically favourable as in the case of semi-local strings [3].

However, it turns out that even in the absence of such topological constraints or symmetry arguments, solutions to non-linear field equations can have localised lumps of energy. A typical example of this is the solitonic breather solution in the sine-Gordon model.

A more curious set of lumps in field theories are those which exist for a long, but finite, time without any of the reasons listed above. These go by the name of oscillons [4] and exist as a localized oscillating state, with the oscillations remaining coherent for much longer than naive dimensional analysis would suggest [5, 6]. Simulations of ϕ^4 scalar theories in two dimensions [7] suggest that they decay only after millions of natural time units whereas in three dimensions they last for thousands of natural time units [8]. The mechanism for their demise follows from their oscillating nature, with the emitted radiation eventually causing oscillons to decay. There is precedence for oscillating solutions to continue for

ever, although it has only been observed in 1+1 dimensions in breathers of the sine-Gordon model. However, this model is rather special in that it is integrable, containing an infinite number of conserved currents. An understanding of the evolution of oscillons was gained in [9] using Fourier analysis to construct strictly periodic solutions, later termed "quasi-breathers" in [10]. Such exactly periodic solutions exist because they include a component of inwardly directed radiation, unlike the oscillon solutions which only radiate outwards, albeit in small amounts. So while quasi-breathers are not expected to be physical their construction shows that the amount of radiation involved is small enough (in some cases) to make them a valid approximation.

In [11, 12] it has been suggested that oscillons cease to exist, in any meaningful sense, for dimensions greater than 5 or 6 depending on the details of the potential. This result was found by assuming a Gaussian profile for oscillons and substituting this into the action in order to generate an effective action. Given the extreme sensitivity of oscillon lifetimes on the initial profile [8, 13] this approach can only yield an indication of what may happen and serves as impetus for our study. We use quasi-breathers in our analysis to show how oscillons behave in various dimensions, including their non-existence in higher dimensions.

It is worthwhile mentioning at this point that oscillons have been observed in the laboratory [14], forming in granular materials placed on a vibrating plate. While these oscillons are not described by a relativistic field theory they are similar enough to bear the same name.

Apart from the challenge of understanding oscillons in themselves, interest derives from the possible impact of these objects if they are generated in large numbers in phase transitions. In particular, if such "coherently oscillating" lumps persist for hundreds or thousands of oscillations, they may play a role during post-inflationary preheating and baryogenesis, and delay the subsequent equilibration and thermalisation.

Oscillons have been extensively studied in recent years, mostly in scalar models, but an oscillon has also been reported in the SU(2)-Higgs model [15]. The very recent work of [10] is a detailed numerical study of oscillons in 3+1 dimensions. Improving on the analysis of [13], the authors propose that generic oscillons are composed of a core quasi-breather of a particular oscillation frequency ω and a number of additional modes, depending on the initial state. As time goes on the oscillon will radiate the excess energy while changing frequency [9], its core scanning through a series of quasi-breathers, eventually approaching a critical ω quasi-breather, after which the oscillon decays. The critical ω corresponds to the lowest energy quasi-breather.

The study in two dimensions of [7] showed that a non-spherical perturbation was not deleterious to the oscillon's evolution and so for our study we restrict our attention to spherical symmetry in D space dimensions. The spatial dimension then simply appears as a parameter in an effective 1+1 spacetime field theory, this allows us to consider non-integer dimensions as a way of observing more clearly how oscillon behaviour depends on D. Using this ansatz, we will perform a mode-decomposition of the quasi-breather in a way similar to [9, 10, 13], and numerically find solutions for each ω, for each choice of spatial dimensions D. Restricting to 3+1 dimensions, it was observed in [9] that there is a particular value of quasi-breather frequency, ω_{crit}, which has minimum energy. Oscillons

with frequencies different from ω_{crit} evolve toward this solution as they radiate energy, decaying once they reach this frequency. We shall see that this behaviour persists in other dimensions, although the time it takes for oscillons to reach ω_{crit} depends sensitively on D. We shall confirm the fact that for $D = 3$ lifetimes are of the order of 10^4 [10], whereas for $D = 2$ oscillons can oscillate for many times 10^6. Obviously, in the limit of $D = 0$ an oscillon is simply a particle oscillating in a well and so is strictly periodic.

These quasi-breather profiles can then be used as initial conditions for real-time simulations in order to find the lifetime of these objects. Also, we will use (non fine-tuned) Gaussian initial conditions as in [10, 13] and study how these oscillons approach the quasi-breather evolution as radiation is emitted. We will see that the oscillon closely follows the quasi-breather, and that the two decay in a similar way.

In section 2 we will set up the model, the quasi-breather ansatz and the mode decomposition we use. In section 3 we will determine the quasi-breather solutions numerically, for a range of ω and D. The real-time evolution of these quasi-breathers is presented in section 4, where we also compare to the evolution of a generic Gaussian initial condition. We will discuss the lifetimes of these objects in section 5 and conclude in section 6.

2. Model, ansatz and mode expansion

We shall study oscillons in scalar ϕ^4 in $D + 1$ dimensions starting with the action

$$S = -\int d\hat{t} d^D \hat{x} \left[\frac{1}{2} \hat{\partial}_\mu \hat{\phi} \hat{\partial}^\mu \hat{\phi} + \frac{1}{2} m^2 \hat{\phi}^2 + \frac{1}{3} \alpha \hat{\phi}^3 + \frac{1}{4} \beta \hat{\phi}^4 \right]. \tag{2.1}$$

From this staring point we restrict the number of free parameters by making the potential a symmetric double well, achieved by imposing

$$\alpha = -3m\sqrt{\beta/2}. \tag{2.2}$$

This gives a potential with extrema at

$$\hat{\phi}_{ext} = 0, \frac{m}{\sqrt{2\beta}}, m\sqrt{\frac{2}{\beta}}. \tag{2.3}$$

It is now possible to perform some rescalings of the co-ordinates and the field

$$\hat{\phi} = \frac{m}{\sqrt{2\beta}} \phi, \quad r = m\hat{r}, \quad t = m\hat{t}, \tag{2.4}$$

and consider only radially symmetric field configurations to find

$$S = -\frac{m^{4-D}}{\beta} A_{D-1} \int dt \, dr \, r^{D-1} \left[-\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_r \phi)^2 + \frac{1}{2}\phi^2 - \frac{1}{2}\phi^3 + \frac{1}{8}\phi^4 \right], \tag{2.5}$$

where $A_{D-1} = 2\pi^{D/2}\Gamma(D/2)$ is the volume of the $D - 1$ sphere which has been integrated out.

A quasi-breather is a periodic solution to the equations of motion, and as such can be expanded in modes with frequencies that are integer multiples of some basic frequency ω,

$$\phi(r,t) = \frac{1}{\sqrt{2}}\phi_0(r) + \sum_{n=1}^{\infty}\phi_n(r)\cos(n\omega t). \qquad (2.6)$$

The $\frac{1}{\sqrt{2}}$ in the first term is purely for convenience later. By integrating the action over a period, $2\pi/\omega$ we have

$$S = -\frac{m^{4-D}\pi}{\omega\beta}A_{D-1}\int dr\; r^{D-1}\left[\left(\sum_{n=0}^{\infty}\frac{1}{2}(\partial_r\phi_n)^2\right) - U_{eff}(\phi_n)\right], \qquad (2.7)$$

$$U_{eff}(\phi_n) = -\frac{1}{2}\phi_0^2 - \frac{1}{2}(1-\omega^2)\phi_1^2 - \frac{1}{2}(1-4\omega^2)\phi_2^2 - \cdots \qquad (2.8)$$

$$+\frac{1}{2}\phi_0^3 + \frac{3}{2}\phi_0\phi_1^2 - \frac{1}{8}\phi_0^4 - \frac{3}{4}\phi_0^2\phi_1^2 - \frac{3}{16}\phi_1^4 + \cdots \;, \qquad (2.9)$$

We now have an effective action in terms of dimensionless parameters r, ω for the dimensionless fields ϕ_n. This allows us to reinterpret it in terms of a particle rolling in "time" r through a space spanned by the $\underline{\phi}$ coordinates under the influence of the potential (2.8). A contour plot of this potential, setting $\phi_2 = \phi_3 = \cdots = 0$ and taking $\frac{1}{2} < \omega < 1$ is shown in figure 1, with a maximum at $(\phi_0, \phi_1) = (1, \sim 1.3)$, minimum at $(1,0)$ and a saddle at $(0,0)$. We shall use this plot to undertand the basic shape of the quasi breather profile.

3. Finding the quasi-breather profile

To find the quasi-breather profile for a particular choice of frequency, ω, we need to solve for the trajectory of a particle moving in U_{eff}. Note however that, because of the r^{D-1} in S, there is a frictional force acting to slow down the particle in analogy to finding the bounce profile in quantum tunnelling [16]. Asymptotically, $r \to \infty$, we require that the particle takes the value $\underline{\phi} = \underline{0}$, corresponding to the scalar field living in the vacuum asymptotically. If we were to ignore those ϕ_n with $n \geq 2$ the problem would be to find the location in

Figure 1: A slice through the effective potential. Maxima are at $(0,0)$, $(1,1.3)$ and there is a minimum at $(1,0)$.

figure 1 where a particle can start at "time" $r = 0$ in order to end at $\underline{\phi} = \underline{0}$ as $r \to \infty$. This is achieved by taking an initial guess at the starting point and observing whether the particle undershoots or overshoots $\underline{\phi} = \underline{0}$, then amending the starting point appropriately. In the actual problem one cannot neglect $\phi_{n\geq 2}$ but the process for finding the values $\phi_0(0)$ and $\phi_1(0)$ is the same. The starting values of $\phi_{n\geq 2}$ is less critical, as can be understood from (2.8), and correspond to adjusting the amount of radiation contained
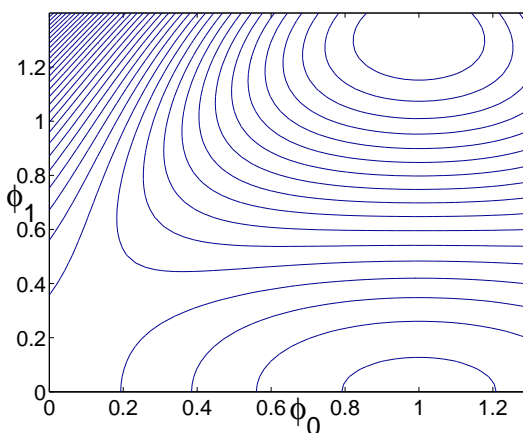
in the quasi-breather solution. Near $\underline{\phi} = \underline{0}$, i.e. the asymptotic region, we see that the effective potential is quadratic with ϕ_n aquiring a mass-squared of $-\frac{1}{2}(1 - n^2\omega^2)$, so those ϕ_n with $n > 1/\omega$ have positive mass-square while those with $n < 1/\omega$ have negative mass-square. This means that the $\phi_{n>1/\omega}$ are oscillatory in the asymptotic region, corresponding to radiation. We shall only concern ourselves with $\frac{1}{2} < \omega < 1$ so that $\phi_{n\geq2}$ correspond to radiation modes. With (2.6) being an expansion of spatially oscillating standing waves, this displays the fact that there is ingoing radiation balancing the outgoing.

3.1 Shooting

A general analytical solution of the equation of motion does not exist, and we will instead proceed to find the quasi-breather profile by numerical means, using a shooting algorithm. Although one can in principle shoot with multiple degrees of freedom we will truncate the mode expansion, using the first four terms, ϕ_n, $n =$0, 1, 2, 3 in our calculations of energy and radiation. Once we have a profile with these components we include the fifth field, ϕ_4, as a consistency check to make sure that it does indeed form only a small contribution.

In the shooting algorithm we pick initial values for ϕ_n, $n < 4$ at $r = 0$, requiring $\phi'_n(r = 0) = 0$ for regularity. As mentioned earlier, the values of the radiation modes, $n =$2, 3, simply dictate how much radiation the quasi breather contains. In [10] the solution is taken which minimizes the amplitude of ϕ_2 asymptotically, here we aim to find the solution which minimizes the total radiation as discussed below. For any particular choice of $\phi_2(0)$ and $\phi_3(0)$ we then vary $\phi_0(0)$ and $\phi_1(0)$ until we find the solution which approaches $\underline{\phi} = 0$ asymptotically. This then gives us the quasi breather for frequency ω and the chosen $\phi_2(0)$, $\phi_3(0)$. By finding the quasi breathers for a range of $\phi_2(0)$, $\phi_3(0)$ we are able to select the one which has the minimum amount of radiation.

An example of a set of profile functions is shown in figure 2 (left); one can see that the amplitudes of the high-n modes decreases, allowing for a valid truncation approximation. As we vary the values of $\phi_2(r = 0)$ and $\phi_3(r = 0)$ in the shooting algorithm we find quasi-
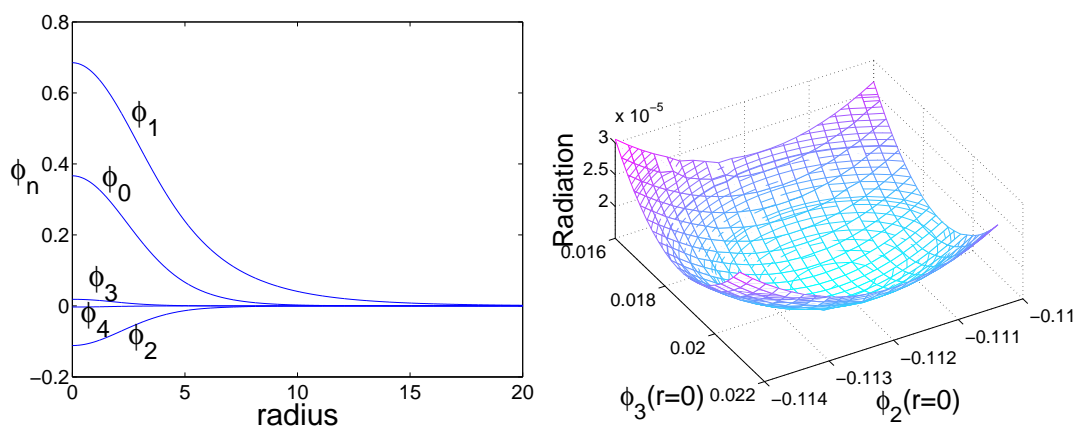

Figure 2: Left: The profiles of ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 for $D = 3$ and $\omega = 0.97$. Right: The dependence of the quasi-breather radiation on $\phi_2(r = 0)$ and $\phi_3(r = 0)$.

breathers with differing amounts of radiation, with one choice giving a minimum, figure 2 (right). We take this solution with minimum radiation to be our quasi-breather.

3.2 Radiation energy

As the oscillon is made by balancing incoming and outgoing radiation then in the realistic situation where there is only outgoing radiation the oscillons will necessarily decay. Here we calculate the rate of energy loss as estimated by constructing a quasi-breather.

First we introduce the standard energy momentum tensor for a real scalar field,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\partial^\rho\phi\partial_\rho\phi + V\right],$$
(3.1)

then we define the momentum flux

$$P^\mu = T^{0\mu},$$
(3.2)

and conservation of the energy-momentum tensor tells us that P^μ is divergence free. By taking P^0 as the energy density we find

$$\dot{E} = -\int d^D x\sqrt{g_{(D)}}\nabla_i P^i,$$
(3.3)

and for a spherically symmetric field we find

$$\dot{E}(R) = -A_{(D-1)}R^{D-1}P^r(r = R),$$
(3.4)

where we have defined $E(R)$ to be the energy within a radius R.

Asymptotically the fields are small so we may approximate the field equation by the Klein-Gordon equation

$$-\ddot{\phi} + \phi'' + \frac{D-1}{r}\phi' = \phi,$$
(3.5)

the closed form solution of which can be approximated asymptotically by the form

$$\phi \sim \cos(kr - \Omega t + \varphi)/r^{(D-1)/2}, \qquad k^2 = \Omega^2 - 1.$$
(3.6)

where we have allowed for a possible phase, φ. Our solution is a sum of waves of different frequencies producing a standing wave between ingoing and outgoing waves,

$$\phi(r \to \infty) \to \frac{1}{r^{(D-1)/2}}\left(f_2\cos(k_2 r + \varphi_2)\cos(2\omega t) + f_3\cos(k_3 r + \varphi_3)\cos(3\omega t) + \cdots\right) \quad (3.7)$$

$$= \frac{1}{2r^{(D-1)/2}}\left(\ f_2\cos(k_2 r + \varphi_2 + 2\omega t) + f_2\cos(k_2 r + \varphi_2 - 2\omega t)\right. \quad (3.8)$$

$$\left. + f_3\cos(k_3 r + \varphi_3 + 3\omega t) + f_3\cos(k_3 r + \varphi_3 - 3\omega t) + \cdots\right),$$

$$= \phi_{in} + \phi_{out},$$
(3.9)

where $k_n^2 = n^2\omega^2 - 1$ and ϕ_{in} contains the terms of the form $\sim \cos(k_n r + \varphi_n + n\omega t)$. Using

$$P^r = \dot{\phi}_{out}\phi'_{out}$$
(3.10)

we can integrate over one period and find that the average rate of energy loss over a period is

$$< \dot{E} > = -\frac{\omega A_{(D-1)}}{8}(2k_2 f_2^2 + 3k_3 f_3^2 + \cdots). \qquad (3.11)$$

For the profiles used in figure 2 we find

$$\dot{E} = 7.9 \times 10^{-6} + 0.31 \times 10^{-6} + 0.035 \times 10^{-6} + \cdots \qquad (3.12)$$

The leading term is from the ϕ_2, with the ϕ_3 and ϕ_4 contributions being sub-dominant.

3.3 Quasi-breather results

We now have everything in place to find a quasi-breather for each given frequency, we shall only focus on frequencies in the range $0.95 \leq \omega \leq 0.99$. The choice of lower bound is because, as we shall see, oscillons spend most of their time above this frequency. Our upper bound is due to numerical inaccuracies in the shooting algorithm giving unreliable results above this frequency. The first result to discuss is the energy of the quasi-breathers as ω varies, then we can understand how oscillons evolve.

In figure 3 we present a plot, for various dimensions, of the quasi-breather energy as a function of ω. In order to get the data for the different dimensions on the same axes we have rescaled each curve so that they meet at $\omega = 0.95$. The different rescaling parameters, $E(D, \omega = 0.95)$, are given in figure 4 showing that to a good approximation the energy of a quasi breather depends exponentially on dimension. In order to calculate the energy of the quasi-breather we note that the asymptotic form of the field (3.6) coupled with the spatial volume element, $\sim r^{D-1}\mathrm{d}r$, causes the energy, $E(R)$, to diverge linearly with R. This divergence is simply due to the radiation, so to define the energy of the quasi-breather we
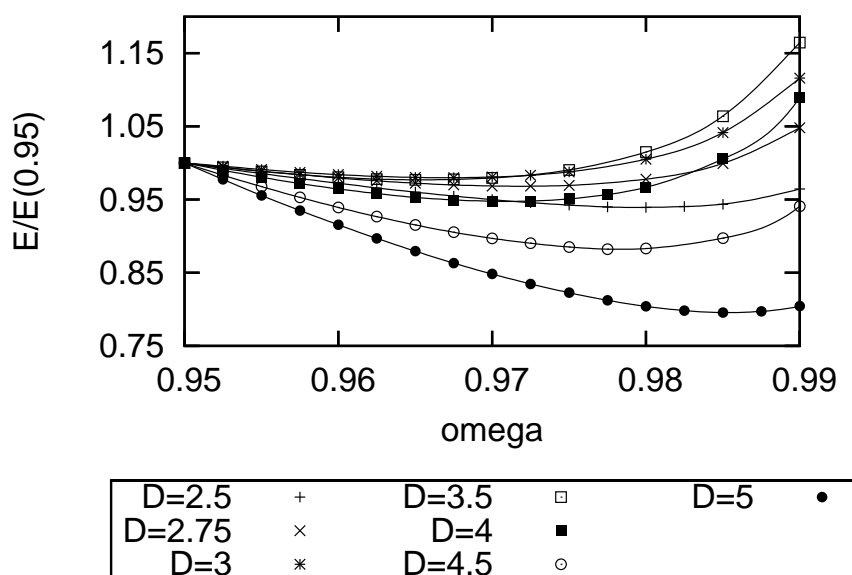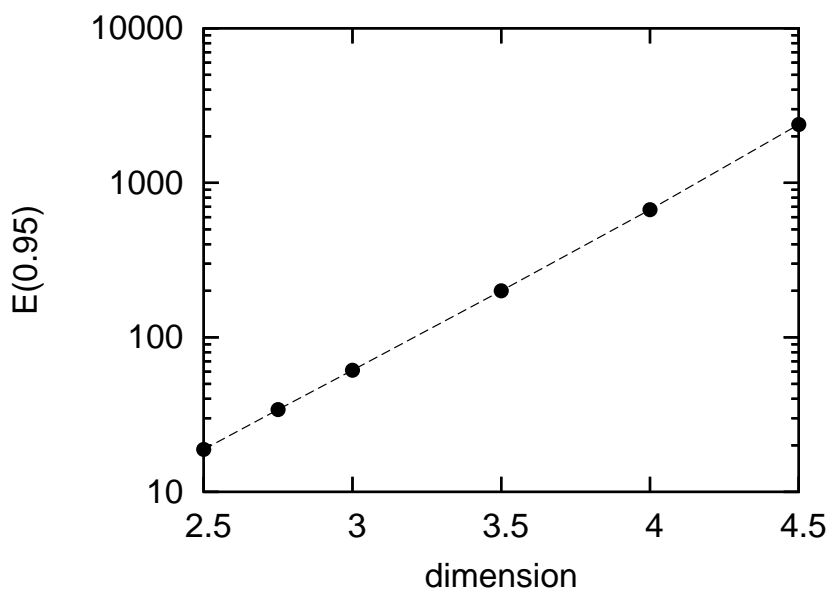
Figure 3: The rescaled energy plot.
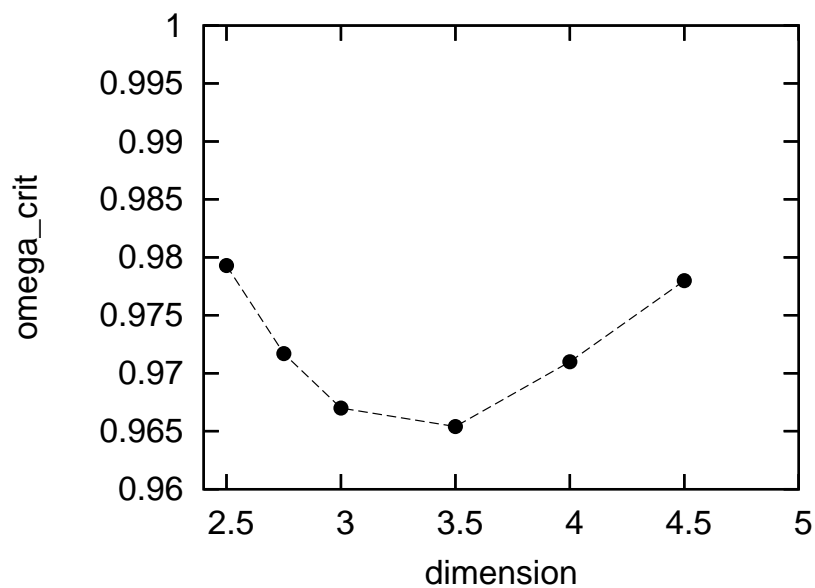
Figure 4: $E(D, \omega = 0.95)$.

Figure 5: The location of the energy minimum.

simply subtract this contribution which is simple to perform once the asymptotic solution has been found.

From figure 3 we see that for each dimension considered the energy shows a minimum at ω_{crit} as ω is varied, with ω_{crit} being dimension dependent. This minimum was first observed in [9] and later in [10]. We can see how this minimum varies with dimension in figure 5. Unfortunately our numerical methods break down below $D \sim 2.5$, because the ϕ_4 mode becomes relevant, and above $D \sim 4.5$, because we could not shoot $\underline{\phi}$ to the vacuum,
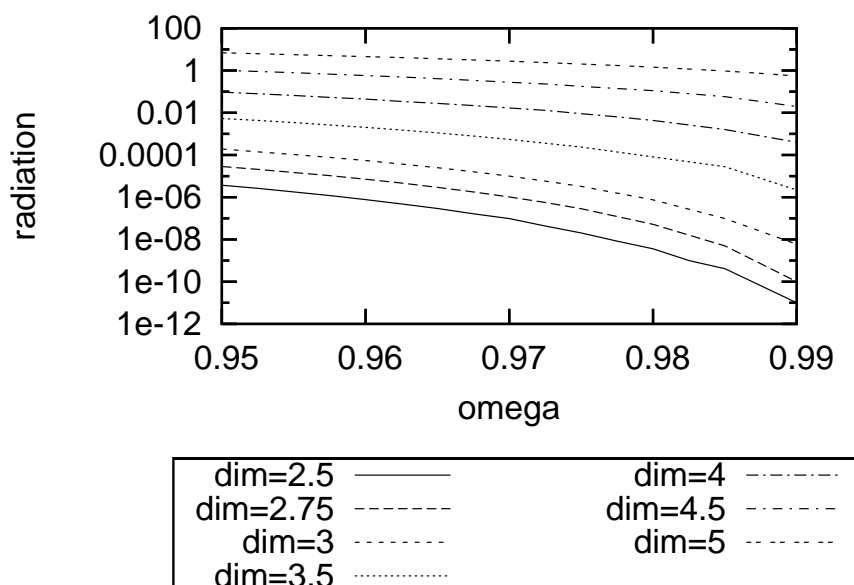
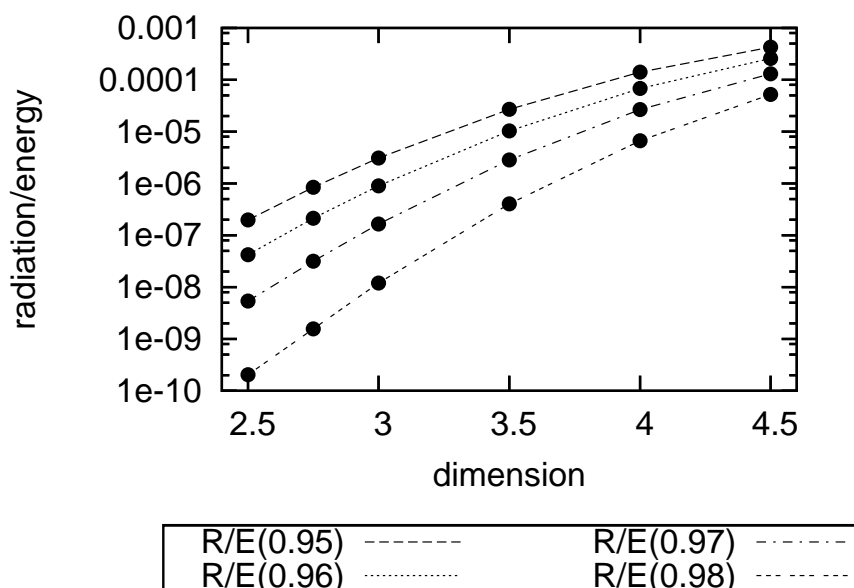**Figure 6:** The radiation of a quasi breather.

Figure 7: The ratio of radiation to energy as a function of dimension.

so we are unable to give a complete plot of the energy minima. It is possible that the techniques advocated in [10] would give a more complete picture, it would be particularly interesting to see how the curve develops below $D = 2.5$.

As explained in [9] oscillons are expected to evolve by radiating energy, and so move on to a quasi breather with lower energy and different ω. Once the quasi-breather with ω_{crit} is reached there is no quasi-breather with lower energy, so the oscillon radiation causes the lump to radiate away completely. We shall see in the next section that this is a good description of the dynamics of oscillons, although we remind the reader that quasi-breathers

are only expected to be an approximation to oscillon evolution owing to the presence of ingoing radiation in a quasi-breather but not in an oscillon.

Along with the calculation of energy for the quasi-breathers we are able to calculate the radiation emitted (and received) as a function of ω for each dimension, this is presented in figure 6. A measure of how long an oscillon can be expected to last comes from the ratio of the rate of energy loss (3.11) to energy and is plotted in figure 7. From this graph we see that as we increase the number of spatial dimensions the amount of radiation relative to energy inside a quasi-breather increases by many orders of magnitude, so reducing the lifetime of an oscillon. In fact, we can derive a more complete version of the history of oscillons using quasi-breathers as we shall see in section 4.1.

4. Real-time evolution

Through a detailed minimisation process, we have now determined the solutions with the lowest radiation for a range of ω and D. The radiation modes correspond to in- and out-going radiation, and if they are both included, the quasi-breather should in principle be stable, possibly up to the order of truncation of the mode expansion.

In physical situations we cannot expect there to be incoming radiation so quasi-breathers are necessarily an approximation to oscillons. Here we test the state of this approximation by comparing the real-time evolution of lumps with initial conditions that are either Gaussian or quasi-breather. The quasi-breather profile would, of course, lead to an exactly periodic evolution so to make the comparison more sensible we damp the radiation at some large radius.

Ultimately, we are interested in lumps of energy created through some finite energy process, such as a phase transition, and so a real-life oscillon is not a quasi-breather. Following [10], we will distinguish between quasi-breathers and oscillons, and represent the latter by Gaussian initial conditions.

We will study the time evolution by discretising the equation of motion, in a simple way. The equation of motion reads, imposing spherical symmetry explicitly,

$$\partial_t \phi_r(t) + \epsilon \Delta(\phi_r(t)) = \Pi_r(t),$$

$$\partial_t \Pi_r(t) + \epsilon \Delta(\Pi_r(t)) = -\Gamma \Pi_r(t) + \partial_r \partial_r^{'} \phi_r(t) - \frac{1}{dr} \left(1 - (1 - dr/r)^{D-1} \right) \partial_r^{'} \phi_r(t)$$

$$- \left(1 - \frac{3}{2} \phi_r(t) + \frac{1}{2} \phi_r^2(t) \right) \phi_r(t), \tag{4.1}$$

with

$$\partial_t(\ldots)_r = \frac{(\ldots)(r, t+dt) - (\ldots)(r,t)}{dt}, \quad \partial_t^{'}(\ldots)_r = \frac{(\ldots)(r,t) - (\ldots)(r, t-dt)}{dt}, \tag{4.2}$$

and similarly for spatial derivatives, and

$$\epsilon \Delta(\ldots) = \epsilon \frac{dr^3}{dt} \left(\partial_r \partial_r^{'} \right)^2 (\ldots). \tag{4.3}$$

We apply this simple leap-frog algorithm with a large lattice (20001 sites), small lattice spacing ($dr = 0.01$), time-step ($dt = 0.005$) and an equally spaced grid with boundary

conditions $\phi(r = 20001) = 0$, $\partial_r \phi(r = 0) = 0$. In [10] it was found that at the level of fine-tuning applied there, the maximum lifetime and the corresponding initial condition depend somewhat on the discretisation. However at the level of precision employed here, this will not be important; we are not attempting to fine-tune to the maximum lifetime.

In order to get rid of the emitted radiation, we mimic an infinitely distant spatial boundary by adding a damping term only beyond a certain radius R_b.

$$\Gamma = 0.1, \quad r > R_b; \quad \Gamma = 0, \quad r < R_b. \tag{4.4}$$

The oscillon core roughly stretches to $r = 30$ and we concluded that $R_b = 60$ and $\Gamma = 0.1$ were reasonable choices. In order to get rid of spurious radiation production on the lattice scale, we also included a Kreiss-Oliger ∂_r^4 damping term in the equations of motion, controlled by a multiplicative parameter ϵ [13, 17].

The initial condition for the oscillon is a Gaussian,

$$\phi(0)_r = A \exp\left(-\frac{r^2}{2\sigma^2}\right), \tag{4.5}$$

which is a two-parameter family of profiles, determined by the amplitude A and the width σ. We will fix $A = 2$, i.e. the center of the oscillon starts out in the potential minimum at $\phi = 2$, while the asymptotic region is in the minimum at $\phi = 0$. We will also fix $\sigma = 3.3$, and consider it a generic initial condition. Not all values of σ give long-lived oscillons, but 3.3 is by no means fine-tuned.

We will compare the evolution of the Gaussian oscillon to the trajectory starting from a quasi-breather.

4.1 Quasi-breathers and oscillons

During the real-time evolution the oscillon is not strictly periodic, nevertheless we can define
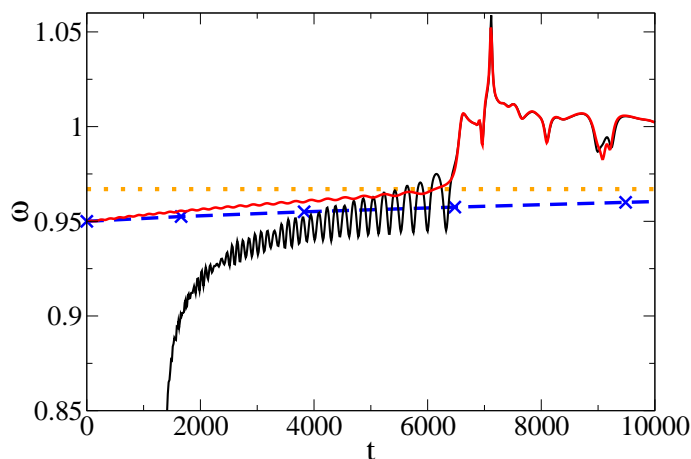

Figure 8: The effective oscillation frequency for a $D = 3$ oscillon with Gaussian initial conditions (black, large oscillations), quasi-breather $\omega = 0.95$ initial conditions (red/grey) and the estimate (4.7) (blue dashed). The orange dotted line represents ω_{crit}.
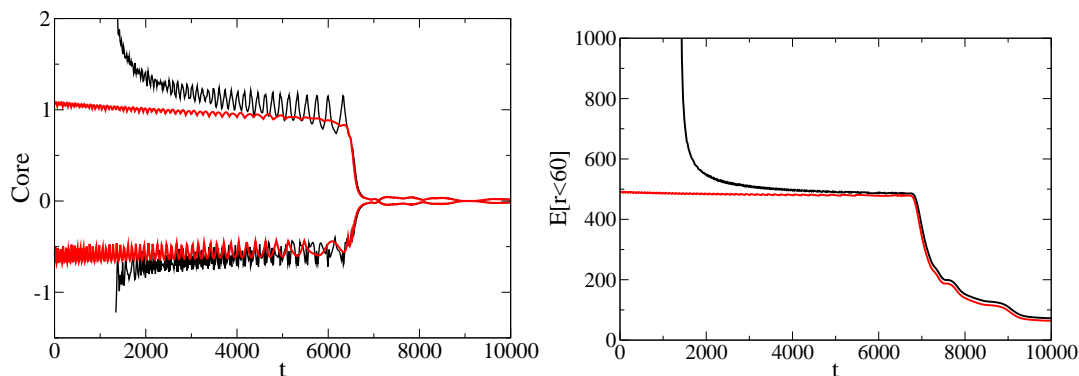
Figure 9: The envelope of the center ($r = 0$) oscillation (left) and the energy within a shell of $r = R_b = 60$ (right). Black lines is the oscillon with Gaussian initial conditions, red/grey using the $\omega = 0.95$ quasi-breather as an initial condition.

an effective frequency of the oscillation by studying the core of the oscillon and defining

$$\omega_{\text{eff}}(t) = \frac{2\pi}{\delta t_{\text{crossing}}}, \tag{4.6}$$

where $\delta t_{\text{crossing}}$ is simply the time between one crossing of $\phi(r = 0)$ through zero and the second crossing after that. This will enable us to measure how the frequency of an oscillon changes in time, as well as finding the frequency at which the oscillon finally decays. Both of these quantities can also be calculated within the quasi-breather picture. The time dependence of ω is found because given the energy $E(\omega)$ (figure 3) we can calculate $dE/d\omega$, so along with (3.11) we find $t(\omega)$ to be

$$t - t_i = \int_{t_i}^{t} dt = \int_{\omega_i}^{\omega} \frac{dE/d\omega}{dE/dt} d\omega. \tag{4.7}$$

Figure 8 presents a comparison in three spatial dimensions of $\omega(t)$ for our three cases: the semi-analytic prediction of 4.7 (blue, dashed); the numerical evolution starting with (non fine tuned) Gaussian initial conditions (black); and the numerical evolution starting with the $\omega_i = 0.95$ quasi-breather profile (red/grey).

The frequency ω_{eff} for the quasi-breather initial conditions slowly scans through ω until it reaches $\omega_{\text{crit}} \simeq 0.967$ where it decays; this happens around $t = 6150$. Eq. (4.7) predicts a longer lifetime by a factor of ~ 3 giving $t \sim 17000$, this over-estimation turns out to be typical and presumably indicates that the quasi-breather picture underestimates the amount of radiation. After the decay, the characteristic frequency becomes 1, the radiation frequency.

The oscillon from Gaussian initial conditions starts out far away from the quasi-breather (the center is initially at $\phi = 2$), but proceeds to shed much of its energy and approach the quasi-breather behaviour. The frequency has a large modulation, but as the critical frequency is reached the oscillon decays in a way remarkably similar to the quasi-breather.

This approach of the case with Gaussian initial conditions to that of the quasi-breather evolution can also be seen by measuring the time development of the core in the lumps,

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figure 9 (left) shows the envelope of the time evolution for the core of the quasi-breather (red/grey) and the Gaussian oscillon (black). The modulation of the oscillon frequency is manifest, but once the decay happens, the two evolve very similarly.

Finally, we plot in figure 9 (right) the energy within a sphere of radius R_b, for the Gaussian oscillon (black) and the quasi-breather (red/grey). As expected, the oscillon sheds its energy out of the ball into radiation (which is then damped away outside the ball). The energy approaches the quasi-breather value, and reaches it just in time for the decay. Note that the decay is delayed by about 600 in time, compared to the crossing of the critical frequency.

Recalling that the initial conditions for the Gaussian were not fine-tuned, this is strong evidence that the quasi-breather behaves as an attractor and is a useful way of understanding oscillon dynamics.

4.2 More than 3 dimensions

As argued in [11, 12] and indicated in figure 7 we should expect that oscillons have rather shorter lives in higher dimensions. The impact of this on the quasi-breather picture is that we should expect it to be less accurate, after all, the quasi-breathers are strictly periodic and have infinite lifetimes. Repeating the analysis of the previous section, but now in four spatial dimensions, we find the results presented in figure 10. Again we note that the lifetime as predicted by the quasi-breather arguments is longer than the measured lifetime, by a factor of ~ 2 (figure 10 (left)). However, by observing the evolution of the core (figure 10 (right)) we see that the oscillon phase ends at around $t \sim 400$ which is where $\omega_{\rm eff}$ hits $\omega_{\rm crit} \sim 0.971$, thus strengthening the argument that oscillons decay once they reach the critical frequency.

4.3 Less than 3 dimensions

Because of concerns about the validity of the mode truncation, we did not trust finding quasi-breathers in $D = 1, 2$; the techniques of [10] may well be more reliable in this regime. Instead we generalised to non-integer D, $D = 2.5, 2.75$ which allows us to see more clearly
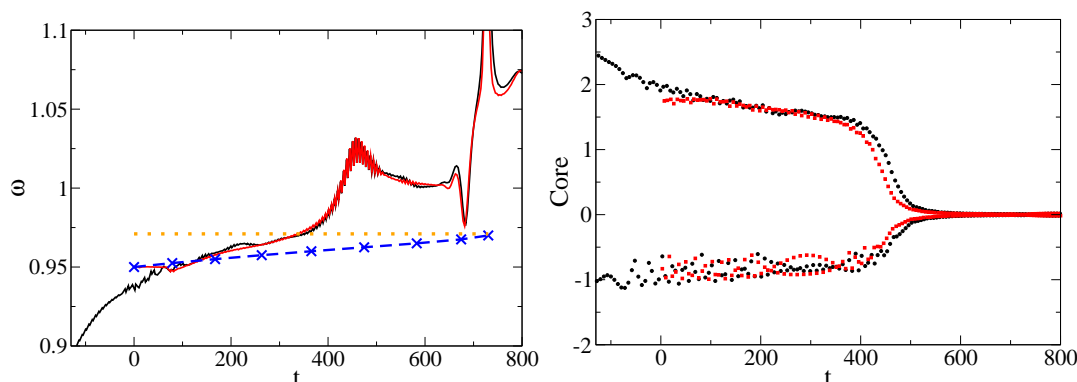

Figure 10: Effective frequency (left) and center envelope (right), for $D = 4$. Color coding as in figure 8.
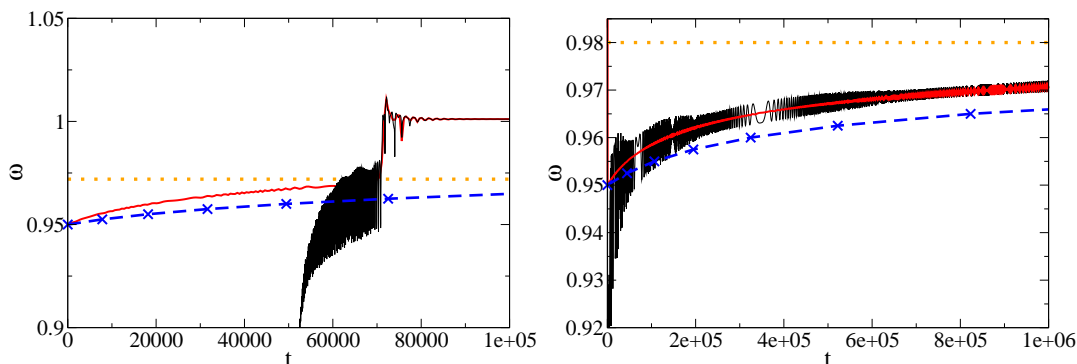
Figure 11: Effective frequency for $D = 2.75$ (left) and $D = 2.5$ (right). Quasi-breather initial conditions (red/grey), Gaussian initial conditions (black), estimate (blue dashed) and critical frequency (orange dotted).

any dependence on spatial dimension. We are able to do this becuase we have assumed spherical symmetry, meaning that the dimension D appears only as a parameter in our evolution equations.

In figure 11 we see the evolution of ω for oscillons with quasi-breather initial conditions, Gaussian initial conditions and the semi-analytic estimate. As expected we see that oscillons last longer in these smaller spatial dimensions, with the semi-analytic reasoning overestimating the lifetime by a factor of a few.

Figure 11 (right) shows $\omega_{\mathrm{eff}}(t)$ for $D = 2.5$. Note that the timescale of interest is now 10^6 and we did not evolve the simulation up to the decay time in this case; $\omega_{\mathrm{crit}} \simeq 0.98$ is far from reached. The Gaussian oscillon (black) closely follows the quasi-breather oscillon from times $\simeq 20000$. The estimate of the evolution (blue dashed) is somewhat off in the same way as the other cases, although the timescale and general trend is well reproduced. The prediction is that the oscillons could live as long as 8×10^6 in natural time units.

4.4 Approaching ω_{crit} from above

So far we have considered only those oscillons whose frequency is less than the critical frequency, and we have seen that as they evolve they follow along one of the curves in figure 3 from some $\omega < \omega_{\mathrm{crit}}$ until they reach $\omega = \omega_{\mathrm{crit}}$ at which point they decay. The evolution of the oscillon as it approaches ω_{crit} is observed to be rather gentle, slowing down because the radiation of a quasi-breather decreases as we approach ω_{crit} from below. Figure 3 reveals another possibility for oscillons, namely those starting from $\omega > \omega_{\mathrm{crit}}$, in this case the quasi-breather picture would say that ω approaches ω_{crit} from above with the oscillon following down one of the curves in figure 3. Using Gaussian initial conditions we were not able to find an oscillon with frequency larger than the critical value, however we are able to start with a quasi-breather profile satisfying $\omega > \omega_{\mathrm{crit}}$. The results are somewhat disappointing for the quasi-breather picture, as the subsequent evolution did not show any oscillon phase. Instead, with the exception of $D = 3$, the field dissipated rapidly straight into radiation, $\omega = 1$, as shown in figure 12.
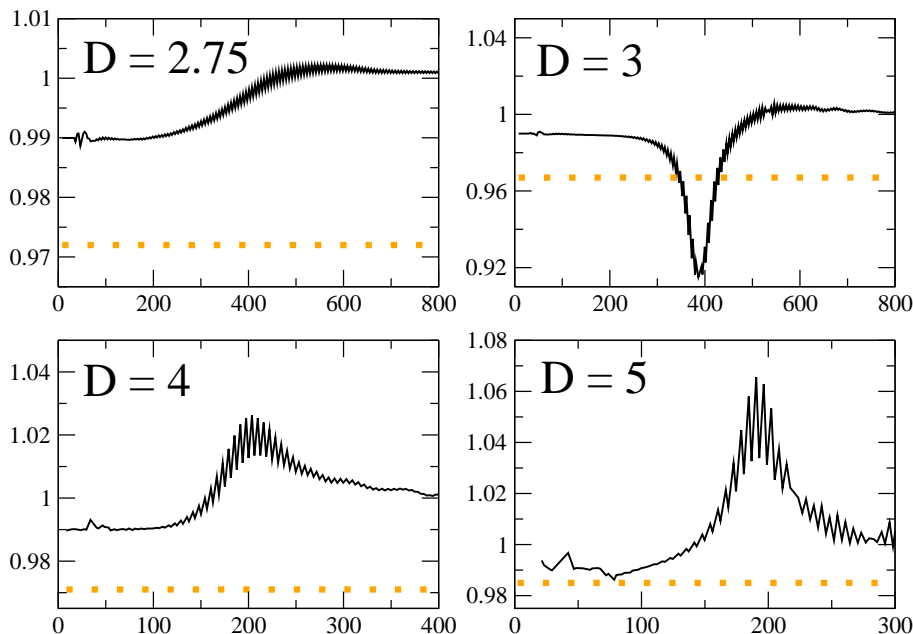
Figure 12: The effective frequency for various D when starting from an $\omega = 0.99$ quasi-breather. The orange dotted line is ω_{crit}.

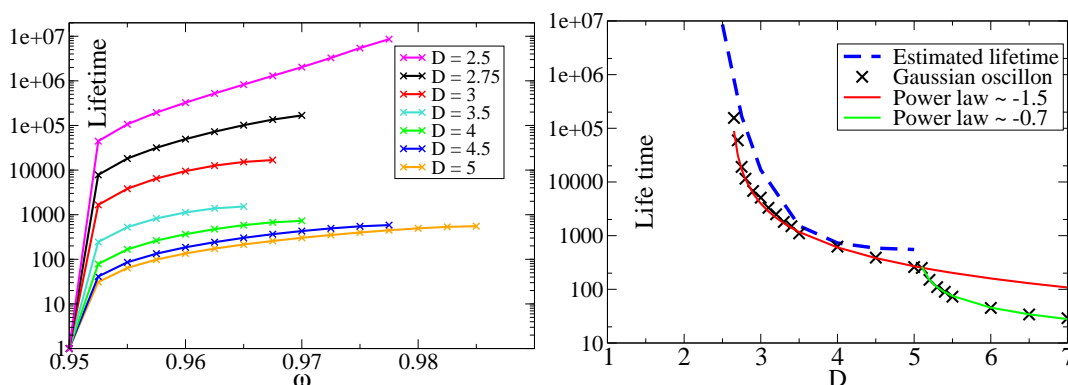## 5. Oscillon life-times and dimensionality

Figure 13: Left: The evolution of ω as predicted by the quasi-breather model using 4.7. Right: Lifetimes for an oscillon starting from a Gaussian with $\sigma = 3.3$ for $2.5 \leq D \leq 7$.

In this section we shall collect together the information we have found about oscillons and quasi-breathers, and show how the quasi-breather picture predicts oscillon evolution should depend on dimension.

We have already seen some specific examples of how the quasi-breather model predicts the evolution of ω using (4.7), in figure 13 (left) we collect together the predictions for a range of dimensions $2\frac{1}{2} \leq D \geq 5$. As noted previously these tend to be off by a factor of a few, overestimating the lifetime, but we can clearly see that the oscillons in lower dimensions take much longer to reach the critical frequency. In higher dimensions we find

that the predicted lifetime is simply not long enough for the system to enter into an oscillon phase.

We may now compare the lifetime for some specific initial condition to that of the quasi-breather oscillon model and see how they compare in different dimensions

As was shown in [10, 13], in D=3 one can tune σ to produce oscillons with a large lifetime, but the level of fine-tuning necessary is extreme. It is then surprising to find that in $D < 3$, oscillons living for $t \simeq 10^6$ are rather more generic and require little tuning of initial conditions. The converse is true in higher dimensions, with the amount of precision required to find a long-lived oscillon increasing with D. This is the main reason that the results of [11, 12], based on substituting an oscillon profile into the action, needed to be confirmed with the more detailed analysis we have presented. Rather than trying to find the fine-tuned initial conditions which maximize the lifetime in each dimension, we shall take the same Gaussian initial conditions ($\sigma = 3.3$, $A = 2$) in the various dimensions and compare the lifetime of the subsequent oscillon with that of the quasi-breather model. The results are plotted in figure 13 (right) and show that the semi-analytic predictions are well matched by the numerical evolution.

In order to guide the eye we have overlaid a curve of the form

$$t_{\text{lifetime}} = \frac{c}{(D_{\text{crit}} - D)^{-a}}, \tag{5.1}$$

with $c = 1000$, $a = -1.5$ and $D_{\text{crit}} = 2.65$, suggesting that the lifetime goes to infinity for some $D > 0$, and possibly for $D > 2$. While this is a possibility, the quasi-breather profile could conspire to have zero amplitude for the radiation modes, we believe that the extrapolation of the quasi-breather results to $D = 2$ require a finite lifetime. In particular, figure 7 does not suggest that the radiation goes to zero at $D = 2$. Note also that there is a kink in the curve of figure 13 (right) at $D \simeq 5$, after which the lifetime suddenly drops (there are now only a handfull of periods), where a similar fit gives $c = 45$, $a = -0.7$ and $D_{\text{crit}} = 5.02$. Although the precise fitting parameters are not essential, the change in power suggests a different regime (no oscillons) has set in.

6. Conclusion

The interpretation of oscillons in terms of quasi-breathers gives a simple and consistent picture of why such long-lived objects should exist and may even be generated in a generic phase transition.

In our setup, the quasi-breathers are the periodic solutions to the equations of motion with minimal outgoing radiation. The presence of this radiation renders the quasi-breather unstable, although since the radiation is minimal, presumably it is the longest lived. Even without excessive fine-tuning, localised lumps (oscillons) can approach and follow the evolution of quasi-breathers, which hence must be considered (at least) weak attractors in the space of classical solutions.

A quasi-breather exists for each choice of frequency. Given an initial choice of ω, the configuration evolves through a sequence of quasi-breathers, approaching a critical

frequency, corresponding to the lowest energy quasi-breather. The time it takes to do so is well reproduced by considering the emitted radiation. As the critical frequency is reached, the energy can no longer be lowered, and the quasi-breather decays completely into radiation.

We performed this analysis for a range of D, the number of spatial dimensions, in order to establish the stability/existence of oscillons in higher dimensions. Using the quasi-breather approach we have provided an understanding for the prescence of very long-lived solutions in low dimensions and short-lived solutions in higher dimensions. Although our method could not cope with $D \leq 2$, it is tempting to conjecture from (5.1) that the lifetime will diverge for some $D > 0$, although the extrapolation of figure 7 does not seem to support this. Perhaps including more radiation modes and more hard work can settle this. While we have studied the dependence on D we have not studied the dependence on $V(\phi)$. One especially interesting case would be the sine-Gordon potential where we know there exist exactly periodic solutions in $D = 1$ [18] as well as $D = 0$, which is just a particle in a well.

For $D > 3$, the lifetimes become smaller and smaller, and oscillons rush to reach the quasi-breather before decaying. For $D > 5$ we were unable to find quasi-breathers with our shooting method, and it indeed seems like a wholly new, oscillon-free, regime sets in. This is where the time a configuration takes to reach a quasi-breather is longer than the lifetime of the quasi-breather itself, so no oscillon can get established.

In a 3+1-dimensional expanding Universe, assuming that the expansion does not significantly influence the field evolution, it is therefore natural to expect oscillons to be generated in phase transitions or during (p)reheating after inflation, with lifetimes of order 10^4 in mass units, but probably not much longer. Scalar fields thermalise on much longer timescales than this [19, 20], but when including gauge degrees of freedom equilibraton is much faster (see for instance [21, 22]). Oscillons may potentially delay this process if they are produced in large numbers, and if the effects of the gauge field do not substantially change the picture.

In order to make further statements about this point, we need to look for quasi-breathers in more realistic settings such as the Standard Model and its extensions. In the restriction to the SU(2)-Higgs model [15], an approach similar to the one employed here should work, in spite of the complexity introduced by the additional degrees of freedom. Work is under way to address this.

Acknowledgments

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